Exponentiation and Logarithms

Review Material for Algorithm Analysis

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Logarithms

- Definition-ish...
  - $\log_B X = Y$ is saying Ⓐ
  - Verbose: $\log_B X$ is asking the question "to what power B must be raised so that we get X?"
  - Example: $\log_2 32$ is asking the question "to what power 2 must have been raised so that we get 32?"
- Notational:
  - $\ln X = \log_e X$; typical with math people
  - $\lg X = \log_2 X$; typical with cs people
  - $\log X = \log_{10} X$; typical with normal people

Exponents

$$X^A X^B = X^{A+B}$$
$$\frac{X^A}{X^B} = X^{A-B}$$
$$X^{A^B} = X^{AB}$$
$$X^N + X^N = 2X^N \neq X^{2N}$$
$$2^N + 2^N = 2^{N+1}$$

Working with Exponentiation and Logarithms

$$\log_b ac = \log_b a + \log_b c$$
$$\log_b \frac{a}{c} = \log_b a - \log_b c$$
$$\log_b a^c = c \log_b a$$
$$\log_b a = \frac{\log_c a}{\log_c b}$$
$$b^{\log_c a} = a^{\log_c b}$$
$$(b^a)^c = b^{ac}$$
$$b^a b^c = b^{a+c}$$
$$\frac{b^a}{b^c} = b^{a-c}$$
$$(ab)^c = a^c a^c$$

$$\lg(2n \lg n) = \lg(2) + \lg(n) + \lg(\lg(n))$$
$$= Ⓐ$$
$$\lg(\frac{n}{2}) = \lg(n) - \lg(2)$$
$$= Ⓐ$$
$$\lg(\sqrt{n}) = \lg(n^{\frac{1}{2}})$$
$$= Ⓐ$$
$$2^{\lg n} = n^{\lg 2}$$
$$= Ⓐ$$

Working with Exponentiation and Logarithms

$$
\begin{aligned}
\log_b ac &= \log_b a + \log_b c \\
\log_b \frac{a}{c} &= \log_b a - \log_b c \\
\log_b a^c &= c \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
b^{\log_c a} &= a^{\log_c b} \\
(b^a)^c &= b^{ac} \\
b^a b^c &= b^{a+c} \\
\frac{b^a}{b^c} &= b^{a-c} \\
(ab)^c &= a^c a^c
\end{aligned}
$$

$$
\begin{aligned}
\lg(\lg(\sqrt{n})) &= \lg(\lg(n^{\frac{1}{2}})) \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
\log_4(n) &= \frac{\lg(n)}{\lg(4)} \\
&= \textcircled{A}
\end{aligned}
$$

Working with Exponentiation and Logarithms

$$
\begin{aligned}
\log_b ac &= \log_b a + \log_b c \\
\log_b \frac{a}{c} &= \log_b a - \log_b c \\
\log_b a^c &= c \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
b^{\log_c a} &= a^{\log_c b} \\
(b^a)^c &= b^{ac} \\
b^a b^c &= b^{a+c} \\
\frac{b^a}{b^c} &= b^{a-c} \\
(ab)^c &= a^c a^c
\end{aligned}
$$

$$
\begin{aligned}
n^2 2^{3\lg(n)} &= n^2 2^{\lg(n^3)} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
\frac{4^n}{2^n} &= \frac{(2^2)^n}{2^n} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A}
\end{aligned}
$$

Working with Exponentiation and Logarithms

$$
\begin{aligned}
\log_b ac &= \log_b a + \log_b c \\
\log_b \frac{a}{c} &= \log_b a - \log_b c \\
\log_b a^c &= c \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
b^{\log_c a} &= a^{\log_c b} \\
(b^a)^c &= b^{ac} \\
b^a b^c &= b^{a+c} \\
\frac{b^a}{b^c} &= b^{a-c} \\
(ab)^c &= a^c a^c
\end{aligned}
$$

$$
\begin{aligned}
\lg(2^n) &= n\lg(2) \\
&= \textcircled{A} \\
4^n &= (2^2)^n \\
&= \textcircled{A} \\
2^{2\lg n} &= (2^{\lg n})^2 \\
&= \textcircled{A} \\
&= \textcircled{A} \\
&= \textcircled{A}
\end{aligned}
$$

How Hypothetical Is This?

- Suppose you want $\log_A B$ but all you can compute is $\log_C B$.
  The book says $\log_A B = \frac{\log_C B}{\log_C A}$.
  Is this true? Should we care?
- Ali writes a sorting program that sorts N numbers in $N \log_2 N$ steps.
- John Barr writes a sorting program that sorts N numbers in $N \log_3 N$ steps. What goes through Ali's mind?

Ali's Thoughts

- $\log_A B = \frac{\log_C B}{\log_C A}$; this is what the book said
- $\log_B N = \frac{\log_2 N}{\log_2 B}$
- $\log_3 N = \frac{\log_2 N}{\log_2 3} = \frac{\log_2 N}{\text{Some constant number between 1 and 2}}$
- If I run my program on a computer that is $\log_2 3$ times faster than John's desktop, then his algorithmic superiority will be destroyed

Justification, The General Case

$$\begin{aligned} \text{Suppose } X &= \log_C B \\ Y &= \log_C A \\ Z &= \log_A B \\ \text{Then } C^X &= B \\ C^Y &= A \\ A^Z &= B \end{aligned}$$

$$\begin{aligned} C^X = B = A^Z &= C^{Y^Z} \\ &= C^{YZ} \\ X &= YZ \\ Z &= \frac{X}{Y} \\ \log_A B &= \frac{\log_C B}{\log_C A} \end{aligned}$$

Justification, A Special Case

- Suppose we have:
  - $\log_B N = K$ meaning $B^K = N$
  - $\log_2 B = C$ meaning $2^C = B$
- Derivation

$$\begin{aligned} N &= B^K \\ &= 2^{C^K} \\ &= 2^{CK} \\ \log_2 N &= \log_2(2^{CK}) \\ &= CK \\ &= (\log_2 B)(\log_B N) \\ \frac{\log_2 N}{\log_2 B} &= \log_B N \\ \log_B N &= \frac{\log_2 N}{\log_2 B} \end{aligned}$$